Hom and tensor product

The localization can also be expressed as a tensor product. We'll give a brief intro to tensor and Hom. First we give a quick review of exactness:

Let $M_0, M_1, \dots, M_n$ be R-modules, with maps between them:

$$M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} M_n.$$

The sequence above is exact at M_i if $\operatorname{im} f_i = \ker f_{i+1}$. The whole sequence is exact if it is exact at each M_i for $0 < i < n$.

A short exact sequence is an exact sequence of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0.$$

In this case, $A = \ker(g)$ and $C = \operatorname{coker}(f) \cong B/A$.

Hom

Def: If M, N are R-modules, then $\operatorname{Hom}_R(M,N)$ is the R-module of homomorphisms $M \to N$.

Ex: $\operatorname{Hom}_R\left(\bigoplus_{i=1}^{n} R, N\right) \cong \bigoplus_{i=1}^{n} N$

Hom is a functor in each of its entries:

Fix an R-module M.

Then Hom $(M, -)$ takes a map $A \to B$ to a map

$$\begin{tikzcd} M \arrow[r] \arrow[dr, dashed] & A \arrow[d] \\ & B \end{tikzcd} \qquad \mathrm{Hom}_R(M, A) \longrightarrow \mathrm{Hom}_R(M, B).$$

Hom $(M, -)$ is <u>left-exact</u>, i.e. if

$0 \to A \to B \to C$ is exact, then so is

$0 \to \mathrm{Hom}(M, A) \to \mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C)$.

Hom $(-, M)$ is also a functor, but given $A \to B$,

$$\begin{tikzcd} A \arrow[r] \arrow[dr, dashed] & B \arrow[d] \\ & M \end{tikzcd}$$

we get a map $\mathrm{Hom}(B, M) \to \mathrm{Hom}(A, M)$

So it "reverses arrows." i.e. it's a <u>contravariant</u> functor.

If $A \to B \to C \to 0$ is exact, then

$0 \to \mathrm{Hom}(C, M) \to \mathrm{Hom}(B, M) \to \mathrm{Hom}(A, M)$ is exact.

Tensor products

If M and N are R-modules,

$M \otimes_R N$ is the R-module generated by elts of the form

$m \otimes n$, w/ $m \in M$, $n \in N$

s.t. it satisfies the following relations:

- If $r \in R$, $(rm) \otimes n = r(m \otimes n) = m \otimes (rn)$

- $(m+m') \otimes n = m \otimes n + m' \otimes n$ and

  $m \otimes (n+n') = m \otimes n + m \otimes n'$.

Caution: In general, elts look like finite sums $\sum_i m_i \otimes n_i$. It's usually difficult to tell if two given elts are equal. In fact, it's often a question what the minimum number of simple tensors required to express an element is.

Ex:

1.) $R \otimes_R M \cong M \cong M \otimes_R R$ and $M \otimes_R N \cong N \otimes_R M$

2.) $R[x_1, \dots, x_m] \otimes_R R[x_{m+1}, \dots, x_n] \cong R[x_1, \dots, x_n]$

3.) $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}[x] \cong \mathbb{Q}[x]$.

4.) $I, J \subseteq R$ ideals $\Rightarrow R/I \otimes_R R/J \cong R/(I+J)$

5.) If M is an R-module, S an R-algebra, then $S \otimes_R M$ is an S-module: $s(t \otimes m) = st \otimes m$.

Tensor product is associative and distributes over direct sum. i.e.

$(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ and

$A \otimes (B \oplus C) \cong (A \otimes B) \oplus (A \otimes C)$.

Universal property

Note that $M \times N \to M \otimes N$ defined $m \times n \mapsto m \otimes n$ (as a function, not a morphism) is bilinear over R.

In other words, for fixed $n \in N$, $m \mapsto m \otimes n$ is an R-module homomorphism, and the same in the other coordinate.

The universal property of tensor product:

If $f : M \times N \to P$ is bilinear, $\exists$ a unique homomorphism

$$\bar{f} : M \otimes N \to P \quad \text{s.t.}$$

$$\begin{tikzcd} M \times N \arrow[r, "\otimes"] \arrow[dr, "f"'] & M \otimes N \arrow[d, dashed, "\bar{f}"] \\ & P \end{tikzcd}$$

commutes.

Remark: Tensor product is a functor:

$$A \longmapsto A \otimes M$$

where $\varphi : A \to B$ goes to $\varphi \otimes \mathrm{id} : A \otimes M \to B \otimes M$.

Claim: Tensor product $- \otimes_R M$ is right exact.

i.e. if

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

is exact, then so is

$$A \otimes_R M \xrightarrow[\alpha \otimes id]{} B \otimes_R M \xrightarrow[\beta \otimes id]{} C \otimes_R M \to 0.$$

Pf: If $c \otimes m \in C \otimes_R M$, then there is some $b \in B$ such that $\beta(b) = c$. Thus, $b \otimes m \mapsto c \otimes m$, so $\beta \otimes id$ is surjective.

To prove exactness at $B \otimes_R M$, we first construct a map

$$B \otimes_R M / \mathrm{im}(\alpha \otimes id) \longrightarrow C \otimes_R M$$

defined: $\overline{b \otimes m} \mapsto \beta(b) \otimes m.$

To show it is well-defined, we need to check that $\overline{0} \mapsto 0$. i.e. if $\sum a \otimes m \in A \otimes M$, then

$$\sum a \otimes m \longmapsto \sum \alpha(a) \otimes m \longmapsto \sum \underbrace{\beta(\alpha(a))}_{= 0} \otimes m = 0.$$

Thus, it is well-defined, and surjective (from above).

To show it's an isomorphism, construct a bilinear map

$$C \times M \longrightarrow B \otimes M / \mathrm{im}(\alpha \otimes id)$$

and apply the universal property. (See HW 2 — make sure to show your bilinear map is well-defined.) $\square$

Right-exactness gives us the following important isomorphism.

Prop: If M is an R-module and $I \subseteq R$ an ideal,

Then $R/I \otimes_R M \cong M/IM$,

where $IM = \{am \mid a \in I, m \in M\} \subseteq M$.

Pf: Consider the short exact sequence

$$0 \to I \to R \to R/I \to 0$$

Tensoring by M yields the exact sequence

$$I \otimes M \xrightarrow[\alpha]{} R \otimes M \xrightarrow[\beta]{} R/I \otimes M \to 0$$

Thus $R \otimes M / \operatorname{im}\alpha \cong R/I \otimes M$.

We know $R \otimes M \cong M$, so $\operatorname{im}\alpha = IM$, and we get the desired isomorphism. $\square$

Ex: $\mathbb{Z}/(n) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}/(n) = 0$.

Localization as tensor product

We can describe the localization of a module by first localizing the ring and then tensoring:

Lemma: The map $U^{-1}R \otimes_R M \to U^{-1}M$ defined $\frac{r}{u} \otimes m \longmapsto \frac{rm}{u}$ is an isomorphism of $(U^{-1}R)$-modules.

(Note: we can construct this map via the universal prop of tensor)

Pf: We'll construct an inverse:

$$\varphi: U^{-1}M \to U^{-1}R \otimes_R M \quad \text{defined}$$

$$\frac{m}{u} \longmapsto \frac{1}{u} \otimes m.$$

Why is this well-defined?

If $\frac{m}{u} = \frac{m'}{u'}$, then $vu'm = vum'$, some $v \in U$.

$$\Rightarrow 1 \otimes vu'm = 1 \otimes vum' \Rightarrow v\left(u'(1 \otimes m)\right) = v\left(u(1 \otimes m')\right)$$

$$\Rightarrow \frac{1}{u} \otimes m = \frac{1}{u'} \otimes m'. \quad \square$$

Since we've written localization as a tensor product, we know that it is right-exact. In fact, it also preserves injections, and thus preserves exact sequences! This is called "flatness":

Def: An R-module F is <u>flat</u> if for every injective R-module map $M \to N$, $F \otimes M \to F \otimes N$ is injective as well.

Prop: $U^{-1}R$ is flat as an R-module. Thus, localization preserves exact sequences.

Pf: Assume $\varphi: M' \to M$ is an injection of R-modules.

We want to show $\underset{\displaystyle U^{-1}M'}{\overset{\displaystyle U^{-1}R \otimes_R M'}{\cong}} \to \underset{\displaystyle U^{-1}M}{\overset{\displaystyle U^{-1}R \otimes_R M}{\cong}}$ is injective.

If $\frac{m'}{u} \mapsto \frac{\varphi(m')}{u} = 0$, then $v\varphi(m') = 0$, some $v \in U$

$$\Rightarrow \varphi(vm') = 0 \Rightarrow vm' = 0$$

$$\Rightarrow \frac{m'}{u} = 0. \square$$

As previously mentioned, there are many properties of modules and rings that we can check by "checking locally". For example, we can check that an element of a module is zero by checking that it's zero in the localization at each max'l ideal:

Lemma: R a ring, M an R-module.

a.) If $a \in M$, then $a = 0 \Leftrightarrow \frac{a}{1} = 0$ in M_m for each max'l ideal $m \subseteq R$

b.) $M = 0 \Leftrightarrow M_m = 0 \quad \forall$ max ideal $m \subseteq R$.

Pf: a.) Let $a \in M$. The <u>annihilator</u> of a is defined

$$\mathrm{Ann}(a) = \{ r \in R \mid ra = 0 \}.$$

Note that $\mathrm{Ann}(a) \subseteq R$ is an ideal.

Then $\frac{a}{1} = 0$ in $M_{\mathfrak{m}} \Leftrightarrow ua = 0$ for some $u \notin \mathfrak{m}$

$\Leftrightarrow \mathrm{Ann}(a) \not\subset \mathfrak{m}$

Thus $\frac{a}{1} = 0$ in $M_{\mathfrak{m}}$ for <u>all</u> max'l $\mathfrak{m}$ $\Leftrightarrow \mathrm{Ann}(a) = R$

$\Leftrightarrow a = 0$ in M.

b.) $M = 0 \Leftrightarrow a = 0 \ \forall a \in M \Leftrightarrow a/_1 = 0$ in all $M_{\mathfrak{m}}$. $\square$

Another thing we can check locally is injectivity and surjectivity:

Cor: If $\varphi: M \to N$ is a map of R-modules, then φ is injective (resp. surjective) iff $\varphi_{\mathfrak{m}}: M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is $\forall$ max'l ideals $\mathfrak{m}$.

Pf: ($\Rightarrow$) follows from flatness.

Conversely, if $\ker(\varphi_{\mathfrak{m}}) \cong (\ker \varphi)_{\mathfrak{m}} = 0 \quad \forall \mathfrak{m}$, then $\ker \varphi = 0$. Similarly w/ cokernel. $\square$